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Hyers–Ulam stability of linear partial differential equations of first order

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ABSTRACT

In this work, we will prove the Hyers–Ulam stability of linear partial differential equations of first order.

1. Introduction

Assume that X is a normed space over a scalar field $\mathbb{K}$ and that I is an open interval. Let $a_0, a_1, \ldots, a_{n-1}$ be fixed elements of $\mathbb{K}$. Assume that for a fixed function $g : I \to X$ and for any n-times differentiable function $y : I \to X$ satisfying the inequality

$$\|y^{(n)}(t) + a_{n-1}\, y^{(n-1)}(t) + \cdots + a_1\, y'(t) + a_0\, y(t) + g(t)\| \le \varepsilon$$

for all $t \in I$ and for a given $\varepsilon > 0$, there exists a function $y_0 : I \to X$ satisfying

$$y_0^{(n)}(t) + a_{n-1}\, y_0^{(n-1)}(t) + \cdots + a_1\, y_0'(t) + a_0 y_0(t) + g(t) = 0$$

and $\|y(t) - y_0(t)\| \le K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression for ε with $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$. Then, we say that the above differential equation has the Hyers–Ulam stability. For more detailed definitions of the Hyers–Ulam stability, we refer the reader to [1–3].

Alsina and Ger [4] investigated the Hyers–Ulam stability of differential equations (see also [5,6]): If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \le \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \le 3\varepsilon$ for any $t \in I$.

The above result has been generalized by many mathematicians (Ref. [7,8]). Recently, the author [9] proved the following theorem concerning the Hyers–Ulam stability of a linear differential equation of first order:

Theorem 1. *Let X be a complex Banach space and let $I = (a, b)$ be an open interval, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$ are arbitrarily given with $a < b$. Assume that $g : I \to \mathbb{C}$ and $h : I \to X$ are continuous functions such that $g(t)$ and $\exp\{\int_a^t g(u)\mathrm{d}u\}h(t)$ are integrable on (a, c) for each $c \in I$. Moreover, suppose $\varphi : I \to [0, \infty)$ is a function such that $\varphi(t)\exp\{\mathrm{Re}(\int_a^t g(u)\mathrm{d}u)\}$ is integrable on I, where $\mathrm{Re}\, z$ denotes the real part of a complex number z. If a continuously differentiable function $y : I \to X$ satisfies the differential inequality*

$$\|y'(t) + g(t)y(t) + h(t)\| \le \varphi(t)$$

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for all $t \in I$, then there exists a unique $x \in X$ such that

$$\left\| y(t) - \exp\left\{-\int_a^t g(u)\mathrm{d}u\right\}\left(x - \int_a^t \exp\left\{\int_a^v g(u)\mathrm{d}u\right\} h(v)\mathrm{d}v\right)\right\| \\ \le \exp\left\{-\mathrm{Re}\left(\int_a^t g(u)\mathrm{d}u\right)\right\}\int_t^b \varphi(v)\exp\left\{\mathrm{Re}\left(\int_a^v g(u)\mathrm{d}u\right)\right\}\mathrm{d}v$$

for every $t \in I$.

Throughout this work, we will denote by $\mathbb{R}^+$ the set of all positive real numbers, i.e., $\mathbb{R}^+ = (0, \infty)$, and by $\mathrm{Re}\, z$ the real part of a complex number z.

In this work, we will prove the Hyers–Ulam stability of first-order linear partial differential equations of the form

$$au_x(x, y) + bu_y(x, y) + g(y)u(x, y) + h(y) = 0 \quad (a \le 0, b > 0) \tag{1}$$

and

$$au_x(x, y) + bu_y(x, y) + g(x)u(x, y) + h(x) = 0 \quad (a > 0, b \le 0) \tag{2}$$

where $g, h : \mathbb{R}^+ \to \mathbb{C}$ are continuous functions satisfying the conditions given in Theorems 2 and 3, respectively.

2. Main results

In the following theorem, we prove the Hyers–Ulam stability of a linear partial differential equation of first order (1).

Theorem 2. *Let $u : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ be a function which has continuous partial derivatives with respect to the first and second variables. Moreover, assume that u satisfies the following inequality:*

$$|au_x(x, y) + bu_y(x, y) + g(y)u(x, y) + h(y)| \le \varepsilon \tag{3}$$

for all $x, y \in \mathbb{R}^+$ and for some $\varepsilon \ge 0$, where $a < 0$, $b > 0$ are constants and $g, h : \mathbb{R}^+ \to \mathbb{C}$ are continuous functions. Furthermore, assume that g, h, and u satisfy

(a) $\int_0^y g(t)\mathrm{d}t$ *exists for all* $y \in \mathbb{R}^+ \cup \{\infty\}$;
(b) $\int_0^y \exp\left\{\frac{1}{b}\int_0^w g(t)\mathrm{d}t\right\} h(w)\mathrm{d}w$ *exists for all* $y \in \mathbb{R}^+ \cup \{\infty\}$;
(c) $\int_0^\infty \exp\left\{\frac{1}{b}\mathrm{Re}\left(\int_0^w g(t)\mathrm{d}t\right)\right\}\mathrm{d}w$ *exists;*
(d) $\lim_{\substack{x\to-\infty\\ y\to+\infty}} u(x, y)$ *exists.*

Then, there exists a unique complex number θ such that

$$\left| u(x, y) - \exp\left\{-\frac{1}{b}\int_0^y g(t)\mathrm{d}t\right\}\cdot\left[\theta - \frac{1}{b}\int_0^y \exp\left\{\frac{1}{b}\int_0^w g(t)\mathrm{d}t\right\} h(w)\mathrm{d}w\right]\right| \\ \le \frac{\varepsilon}{b}\exp\left\{-\frac{1}{b}\mathrm{Re}\left(\int_0^y g(t)\mathrm{d}t\right)\right\}\int_y^\infty \exp\left\{\frac{1}{b}\mathrm{Re}\left(\int_0^w g(t)\mathrm{d}t\right)\right\}\mathrm{d}w \tag{4}$$

for all $x, y \in \mathbb{R}^+$.

Proof. We first introduce new coordinates (ξ, η) by a suitable change of axes:

$$\xi = x - \frac{a}{b}y \quad \text{and} \quad \eta = \frac{1}{b}y. \tag{5}$$

If we define $\tilde{u}(\xi, \eta) = u(\xi + a\eta, b\eta) = u(x, y)$, then it follows from (5) that

$$u_x(x, y) = \tilde{u}_\xi(\xi, \eta)\frac{\partial\xi}{\partial x} + \tilde{u}_\eta(\xi, \eta)\frac{\partial\eta}{\partial x} = \tilde{u}_\xi(\xi, \eta),$$
$$u_y(x, y) = \tilde{u}_\xi(\xi, \eta)\frac{\partial\xi}{\partial y} + \tilde{u}_\eta(\xi, \eta)\frac{\partial\eta}{\partial y} = -\frac{a}{b}\tilde{u}_\xi(\xi, \eta) + \frac{1}{b}\tilde{u}_\eta(\xi, \eta).$$

Hence, we have

$$au_x(x, y) + bu_y(x, y) = \tilde{u}_\eta(\xi, \eta),$$

and if we apply this equality to (3), we get

$$|\tilde{u}_\eta(\xi, \eta) + \tilde{g}(\eta)\tilde{u}(\xi, \eta) + \tilde{h}(\eta)| \le \varepsilon \tag{6}$$

for all $\xi, \eta \in \mathbb{R}^+$, where we define $\tilde{g}(\eta) = g(b\eta) = g(y)$ and $\tilde{h}(\eta) = h(b\eta) = h(y)$.

If we set

$$\xi = s - \frac{a}{b}t \quad \text{and} \quad \mu = \frac{1}{b}t, \tag{7}$$

then $\tilde{g}(\mu) = g(b\mu) = g(t)$ and it follows from (a) that

$$\int_0^y \tilde{g}(\mu)\mathrm{d}\mu = \frac{1}{b}\int_0^{by} g(t)\mathrm{d}t \quad \text{exists for any } y \in \mathbb{R}^+. \tag{8}$$

Moreover, if we set

$$\xi = v - \frac{a}{b}w \quad \text{and} \quad \nu = \frac{1}{b}w, \tag{9}$$

then we have $\tilde{h}(\nu) = h(b\nu) = h(w)$ and it follows from (8) and (9) that

$$\int_0^y \exp\left\{\int_0^\nu \tilde{g}(\mu)\mathrm{d}\mu\right\}\tilde{h}(\nu)\mathrm{d}\nu = \frac{1}{b}\int_0^{by} \exp\left\{\frac{1}{b}\int_0^w g(t)\mathrm{d}t\right\} h(w)\mathrm{d}w.$$

Hence, it follows from (b) that

$$\int_0^y \exp\left\{\int_0^\nu \tilde{g}(\mu)\mathrm{d}\mu\right\}\tilde{h}(\nu)\mathrm{d}\nu \quad \text{exists for all } y \in \mathbb{R}^+. \tag{10}$$

Analogously, it follows from (8) and (9) and (c) that

$$\int_0^\infty \varepsilon \exp\left\{\mathrm{Re}\left(\int_0^\nu \tilde{g}(\mu)\mathrm{d}\mu\right)\right\}\mathrm{d}\nu = \frac{\varepsilon}{b}\int_0^\infty \exp\left\{\frac{1}{b}\mathrm{Re}\left(\int_0^w g(t)\mathrm{d}t\right)\right\}\mathrm{d}w \quad \text{exists.} \tag{11}$$

In view of the inequality (6), the conditions (8), (10) and (11), together with Theorem 1, imply that for each fixed $\xi \in \mathbb{R}^+$, there exists a unique complex number $\theta(\xi)$ such that

$$\begin{aligned}&\left|\tilde{u}(\xi,\eta) - \exp\left\{-\int_0^\eta \tilde{g}(\mu)\mathrm{d}\mu\right\}\cdot\left[\theta(\xi) - \int_0^\eta \exp\left\{\int_0^\nu \tilde{g}(\mu)\mathrm{d}\mu\right\}\tilde{h}(\nu)\mathrm{d}\nu\right]\right|\\ &\quad \le \varepsilon\exp\left\{-\mathrm{Re}\left(\int_0^\eta \tilde{g}(\mu)\mathrm{d}\mu\right)\right\}\int_\eta^\infty \exp\left\{\mathrm{Re}\left(\int_0^\nu \tilde{g}(\mu)\mathrm{d}\mu\right)\right\}\mathrm{d}\nu\end{aligned} \tag{12}$$

for all $\eta \in \mathbb{R}^+$.

According to a formula in the proof of [9, Theorem 1], it follows from (6) that

$$z(\eta) = \exp\left\{\int_0^\eta \tilde{g}(\mu)\mathrm{d}\mu\right\}\tilde{u}(\xi,\eta) + \int_0^\eta \exp\left\{\int_0^\nu \tilde{g}(\mu)\mathrm{d}\mu\right\}\tilde{h}(\nu)\mathrm{d}\nu,$$

and in view of (8) and (9), (a), (b), and (d), we conclude that

$$\begin{aligned}\theta(\xi) &= \lim_{\eta\to\infty} z(\eta)\\ &= \lim_{\eta\to\infty}\exp\left\{\frac{1}{b}\int_0^{b\eta} g(t)\mathrm{d}t\right\}u(\xi + a\eta, b\eta) + \lim_{\eta\to\infty}\frac{1}{b}\int_0^{b\eta}\exp\left\{\frac{1}{b}\int_0^w g(t)\mathrm{d}t\right\}h(w)\mathrm{d}w\end{aligned}$$

is a constant, say simply θ.

We know that $\tilde{u}(\xi,\eta) = u(x,y)$ and it moreover follows from (5), (8) and (9) that

$$\int_0^\eta \tilde{g}(\mu)\mathrm{d}\mu = \frac{1}{b}\int_0^y g(t)\mathrm{d}t \quad \text{and} \quad \int_0^\nu \tilde{g}(\mu)\mathrm{d}\mu = \frac{1}{b}\int_0^w g(t)\mathrm{d}t.$$

Analogously, it follows from (9) that $\tilde{h}(\nu) = h(b\nu) = h(w)$. Hence, applying the above arguments to the inequality (12) and taking $y = b\eta$ and $w = b\nu$ into account, we obtain the inequality (4). □

Remark 1. We can show by a tedious calculation that

$$u(x,y) = \exp\left\{-\frac{1}{b}\int_0^y g(t)\mathrm{d}t\right\}\left[\theta - \frac{1}{b}\int_0^y \exp\left\{\frac{1}{b}\int_0^w g(t)\mathrm{d}t\right\}h(w)\mathrm{d}w\right]$$

is a solution of the partial differential equation (1).

Analogously to Theorem 2, we will investigate the Hyers–Ulam stability of the partial differential equation (2).

Theorem 3. *Let* $u : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ *be a function which has continuous partial derivatives with respect to the first and second variables. Moreover, assume that* u *satisfies the following inequality:*

$$|au_x(x, y) + bu_y(x, y) + g(x)u(x, y) + h(x)| \le \varepsilon \tag{13}$$

for all $x, y \in \mathbb{R}^+$ *and for some* $\varepsilon \ge 0$*, where* $a > 0$*,* $b < 0$ *are constants and* $g, h : \mathbb{R}^+ \to \mathbb{C}$ *are continuous functions. Furthermore, assume that* g*,* h*, and* u *satisfy*

(a′) $\int_0^y g(t)\mathrm{d}t$ *exists for all* $y \in \mathbb{R}^+ \cup \{\infty\}$*;*
(b′) $\int_0^y \exp\left\{\frac{1}{a}\int_0^w g(t)\mathrm{d}t\right\} h(w)\mathrm{d}w$ *exists for all* $y \in \mathbb{R}^+ \cup \{\infty\}$*;*
(c′) $\int_0^\infty \exp\left\{\frac{1}{a}\,\mathrm{Re}\left(\int_0^w g(t)\mathrm{d}t\right)\right\}\mathrm{d}w$ *exists;*
(d′) $\lim_{\substack{x\to+\infty\\ y\to-\infty}} u(x, y)$ *exists.*

Then, there exists a unique complex number θ *such that*

$$\begin{aligned}&\left|u(x, y) - \exp\left\{-\frac{1}{a}\int_0^x g(t)\mathrm{d}t\right\}\cdot\left[\theta - \frac{1}{a}\int_0^x \exp\left\{\frac{1}{a}\int_0^w g(t)\mathrm{d}t\right\} h(w)\mathrm{d}w\right]\right|\\ &\quad\le \frac{\varepsilon}{a}\exp\left\{-\frac{1}{a}\mathrm{Re}\left(\int_0^x g(t)\mathrm{d}t\right)\right\}\int_x^\infty \exp\left\{\frac{1}{a}\mathrm{Re}\left(\int_0^w g(t)\mathrm{d}t\right)\right\}\mathrm{d}w\end{aligned} \tag{14}$$

for all $x, y \in \mathbb{R}^+$*.*

Proof. If we set $v(x, y) = u(y, x)$ for all $x, y \in \mathbb{R}^+$, then we have

$$\begin{aligned}u_x(x, y) &= \lim_{h\to 0}\frac{u(x + h, y) - u(x, y)}{h} = \lim_{h\to 0}\frac{v(y, x + h) - v(y, x)}{h} = v_y(y, x),\\ u_y(x, y) &= \lim_{h\to 0}\frac{u(x, y + h) - u(x, y)}{h} = \lim_{h\to 0}\frac{v(y + h, x) - v(y, x)}{h} = v_x(y, x).\end{aligned}$$

So, we obtain

$$au_x(x, y) + bu_y(x, y) + g(x)u(x, y) + h(x) = av_y(y, x) + bv_x(y, x) + g(x)v(y, x) + h(x)$$

and it follows from (13) that

$$|bv_x(y, x) + av_y(y, x) + g(x)v(y, x) + h(x)| \le \varepsilon$$

for any $x, y \in \mathbb{R}^+$. If we exchange the roles of x and y in the above inequality, then we get

$$|bv_x(x, y) + av_y(x, y) + g(y)v(x, y) + h(y)| \le \varepsilon$$

for all $x, y \in \mathbb{R}^+$.

In view of (a′)–(d′), and Theorem 2, there exists a unique complex number θ such that

$$\begin{aligned}&\left|v(x, y) - \exp\left\{-\frac{1}{a}\int_0^y g(t)\mathrm{d}t\right\}\cdot\left[\theta - \frac{1}{a}\int_0^y \exp\left\{\frac{1}{a}\int_0^w g(t)\mathrm{d}t\right\} h(w)\mathrm{d}w\right]\right|\\ &\quad\le \frac{\varepsilon}{a}\exp\left\{-\frac{1}{a}\mathrm{Re}\left(\int_0^y g(t)\mathrm{d}t\right)\right\}\int_y^\infty \exp\left\{\frac{1}{a}\mathrm{Re}\left(\int_0^w g(t)\mathrm{d}t\right)\right\}\mathrm{d}w\end{aligned}$$

for any $x, y \in \mathbb{R}^+$. By exchanging the roles of x and y in the above inequality, we can easily verify the validity of inequality (14). □

Remark 2. By a tedious calculation we can show that

$$u(x, y) = \exp\left\{-\frac{1}{a}\int_0^x g(t)\mathrm{d}t\right\}\left[\theta - \frac{1}{a}\int_0^x \exp\left\{\frac{1}{a}\int_0^w g(t)\mathrm{d}t\right\} h(w)\mathrm{d}w\right]$$

is a solution of the partial differential equation (2).

Remark 3. When the coefficient functions g and h are constants, the Hyers–Ulam stability of (1) or (2) was proved in [10]. But it is an open question whether the Hyers–Ulam stability is still true if the coefficient functions g and h in (1) or (2) are functions of two variables.

3. An example

Let $u : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a given function which has continuous partial derivatives with respect to the first and second variables. For a given constant $c > 0$, we define a continuous function $g : \mathbb{R}^+ \to \mathbb{R}$ by

$$g(t) = \begin{cases} -c & (0 < t \le 1), \\ -ct & (t > 1). \end{cases}$$

Assume that u satisfies

$$|au_x(x, y) + bu_y(x, y) + g(y)u(x, y) + ky| \le \varepsilon$$

for all $x, y \in \mathbb{R}^+$ and for some $\varepsilon \ge 0$, where $a \le 0$, $b > 0$, $k \ge 0$ are constants. If we set $h(y) = ky$, then we have

$$\int_0^y g(t)\mathrm{d}t = \begin{cases} -cy & (\text{for } 0 < y \le 1) \\ -\frac{c}{2}(1 + y^2) & (\text{for } y > 1), \end{cases}$$

$$\int_0^y \exp\left\{\frac{1}{b}\int_0^w g(t)\mathrm{d}t\right\} h(w)\mathrm{d}w = \begin{cases} \frac{b^2k}{c^2} - \frac{bk}{c}\exp\left\{-\frac{c}{b}y\right\}\left(y + \frac{b}{c}\right) & (\text{for } 0 < y \le 1) \\ \frac{b^2k}{c^2}\left(1 - \exp\left\{-\frac{c}{b}\right\}\right) - \frac{bk}{c}\exp\left\{-\frac{c}{2b}\left(1 + y^2\right)\right\} & (\text{for } y > 1) \end{cases}$$

and

$$\begin{aligned} \int_0^\infty \exp\left\{\frac{1}{b}\mathrm{Re}\left(\int_0^w g(t)\mathrm{d}t\right)\right\}\mathrm{d}w &= \frac{b}{c}\left(1 - \exp\left\{-\frac{c}{b}\right\}\right) + \int_1^\infty \exp\left\{-\frac{c}{2b}\left(1 + w^2\right)\right\}\mathrm{d}w \\ &< \infty. \end{aligned}$$

According to Theorem 2, there exists a unique function $\theta : \mathbb{R}^+ \to \mathbb{C}$ such that

$$\begin{aligned} &\left|u(x, y) - \left[\theta\left(x - \frac{a}{b}y\right) - \frac{bk}{c^2}\right]\exp\left\{\frac{c}{b}y\right\} - \frac{k}{c}\left(y + \frac{b}{c}\right)\right| \\ &\quad \le \frac{\varepsilon}{c}\left(1 - \exp\left\{\frac{c}{b}(y - 1)\right\}\right) + \frac{\varepsilon}{b}\exp\left\{\frac{c}{b}y\right\}\int_1^\infty \exp\left\{-\frac{c}{2b}\left(1 + w^2\right)\right\}\mathrm{d}w, \end{aligned}$$

for all $x \in \mathbb{R}^+$ and $0 < y \le 1$, and

$$\begin{aligned} &\left|u(x, y) - \left[\theta\left(x - \frac{a}{b}y\right) - \frac{bk}{c^2}\left(1 - \exp\left\{-\frac{c}{b}\right\}\right)\right]\exp\left\{\frac{c}{2b}\left(1 + y^2\right)\right\} - \frac{k}{c}\right| \\ &\quad \le \frac{\varepsilon}{b}\exp\left\{\frac{c}{2b}\left(1 + y^2\right)\right\}\int_y^\infty \exp\left\{-\frac{c}{2b}\left(1 + w^2\right)\right\}\mathrm{d}w \end{aligned}$$

for all $x \in \mathbb{R}^+$ and $y > 1$.

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